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Non-homogeneous Steady-State Heat Conduction Problem in a Thin Circular Plate and Its Thermal Stresses

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Abstract The present paper deals with the determination of the displacement and thermal stresses in a thin circular plate defined as $0 \leq r \leq b, 0 \leq z \leq h$ under a steady temperature field, due to a constant rate of heat generation within it. A thin circular plate is insulated at the fixed circular boundary $(r = b)$, and the remaining boundary surfaces $(z = 0, z = h)$ are kept at zero temperature. The governing heat conduction equation has been solved by using an integral transform technique. The results are obtained in series form in terms of modified Bessel functions. The results for displacement and stresses have been computed numerically and are illustrated graphically.

Keywords Constant heat generation · Displacement function · Steady state · Thermal stresses · Thermoelastic problem

1 Introduction

During the second half of the twentieth century, non-isothermal problems of the theory of elasticity became increasingly important. This is due to their wide application in diverse fields. The high velocities of modern aircraft give rise to aerodynamic heating, which produces intense thermal stresses that reduce the strength of the aircraft structure.

Nowacki [1] has determined steady-state thermal stresses in a thin circular plate subjected to an axisymmetric temperature distribution on the upper face with the lower

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face at zero temperature and a circular edge. Roy Choudhary [2] has succeeded in determining the quasi-static thermal stresses in a thin circular plate subjected to a transient temperature along the circumference of a circle over the upper face with the lower face at zero temperature and a fixed circular edge thermally insulated. Kulkarni and Deshmukh [3] have determined quasi-static thermal stresses in a steady-state thick circular plate. Deshmukh et al. [4] have determined the quasi-static thermal stresses due to heat generation in a thin hollow circular disk.

In this paper the two-dimensional non-homogeneous steady-state boundary value problem of heat conduction is considered, and the thermoelasticity of a thin circular plate defined as $0 \le r \le b$, $0 \le z \le h$ is studied. This problem deals with the determination of the displacement and thermal stresses under a steady-state temperature field due to a constant rate of heat generation within it. A thin circular plate is insulated at the fixed circular boundary ($r = b$), and the remaining surfaces($z = 0, z = h$) are kept at zero temperature. The governing heat conduction equation has been solved by using an integral transform technique.

The results are obtained in series form in terms of modified Bessel functions. The results for displacement and stresses have been computed numerically and are illustrated graphically. It is believed that this particular problem has not been previously considered.

The results presented here will be useful in engineering problems, particularly in aerospace engineering for stations of a missile body not influenced by nose tapering. The missile skill material is assumed to have physical properties independent of temperature, so that the temperature $T(r, z)$ is a function of radius and thickness only.

2 Formulation of the Problem

Consider a thin circular plate under a steady-state temperature field of thickness h occupying space D defined by $0 \le r \le b, 0 \le z \le h$ due to heat generation at a constant rate of g_0 (in $\mathrm{W \cdot m^{-3}}$) within the solid. The zero heat flux is applied on the fixed circular boundary ($r = b$). The lower surface ($z = 0$) and the upper surface ($z = h$) are at zero temperature. Under these conditions, the thermoelasticity in a thin circular plate due to constant heat generation is required to be determined.

Following Deshmukh et.al. [4], we assume that a circular plate of small thickness h is in a planar state of stress. In fact, "the smaller the thickness of the circular plate compared to its diameter, the nearer to a planar state of stress is the actual state." The displacement equations of thermoelasticity have the form,

$$U_{i,kk} + \left(\frac{1+\nu}{1-\nu}\right) e_{,i} = 2\left(\frac{1+\nu}{1-\nu}\right) a_t T_{,i} \tag{1}$$

$$e = U_{k,k}; \quad k, i = 1, 2, \tag{2}$$

where U_i is the displacement component, e is the dilatation, T is the temperature, and ν and a_t are, respectively, Poisson's ratio and the linear coefficient of thermal expansion of the circular plate material.

Introducing

$$U_i = \psi_{,i}, \quad i = 1, 2,$$

we have

$$\nabla_1^2 \psi = (1 + \nu) a_t T \tag{3}$$

$$\nabla_1^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

$$\sigma_{ij} = 2\mu \left(\psi_{,ij} - \delta_{ij} \psi_{,kk} \right), \quad i, j, k = 1, 2, \tag{4}$$

where μ is the Lamé constant and δ_{ij} is the Kronecker delta symbol.

In the axially symmetric case

$$\psi = \psi(r, z), \quad T = T(r, z)$$

and the differential equation governing the displacement potential function $\psi(r, z)$ is expressed as

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} = (1 + \nu) a_t T \tag{5}$$

The stress functions σ_{rr} and $\sigma_{\theta\theta}$ are given by

$$\sigma_{rr} = \frac{-2\mu}{r} \frac{\partial \psi}{\partial r} \tag{6}$$

$$\sigma_{\theta\theta} = -2\mu \frac{\partial^2 \psi}{\partial r^2} \tag{7}$$

with $\psi = \sigma_{rr} = \sigma_{\theta\theta} = 0$ at $z = 0$ and $z = h$

Also, in the planar state of stress within the circular plate,

$$\sigma_{rz} = \sigma_{zz} = \sigma_{\theta z} = 0 \tag{8}$$

The steady-state temperature of the plate $T(r, z)$ satisfies the following model:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \frac{g_0}{k} = 0 \quad \text{in } 0 \le r \le b, 0 \le z \le h \tag{9}$$

with the boundary conditions,

$$\tfrac{\partial T}{\partial r} = 0 \quad \text{at } r = b, \quad t > 0 \tag{10}$$

$$T = 0 \quad \text{at } z = 0, \quad t > 0 \tag{11}$$

$$T = 0 \quad \text{at } z = h, \quad t > 0 \tag{12}$$

where k and α are the thermal conductivity and thermal diffusivity of the material of the circular plate, respectively. Equations 1–12 constitute a mathematical formulation of the problem.

3 Solution

We first reduce Poisson's Eq. 9 to the Laplace equation by defining a new dependent variable as described below.

A new dependent variable $\theta(r, z)$ is defined as

$$T(r, z) = \theta(r, z) + P(r, z) \tag{13}$$

where the $P(r, z)$ function in the cylindrical co-ordinate system is

$$P(r, z) = -\frac{g_0}{k}\frac{r^2}{4}$$

Then Eq. 13 becomes

$$T(r, z) = \theta(r, z) - \frac{g_0}{k}\frac{r^2}{4} \tag{14}$$

Substituting Eq. 14 into Eqs. 9–12, one obtains the Laplace equation with one inhomogeneous boundary condition,

$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{\partial^2\theta}{\partial z^2} = 0 \tag{15}$$

with

$$\frac{\partial\theta}{\partial r} = \frac{g_0 r}{2k} \quad \text{at } r = b \tag{16}$$

$$\theta = 0 \quad \text{at } z = 0 \tag{17}$$

$$\theta = 0 \quad \text{at } z = h \tag{18}$$

To obtain the expression of the function $\theta(r, z)$, we develop the finite Fourier transform and its inverse transform over the variable z in the range $0 \leq z \leq h$ defined in [5] as

$$\overline{\theta}(\alpha_n, r) = \int_{z'=0}^{h} K\left(\alpha_n, z'\right)\theta\left(r, z'\right)\mathrm{d}z' \tag{19}$$

$$\theta(r, z) = \sum_{n=1}^{\infty} K(\alpha_n, z)\,\overline{\theta}(\alpha_n, r) \tag{20}$$

where

$$K\,(\alpha_n, z) = \sqrt{\frac{2}{h}}\sin\,(\alpha_n z)$$

and $\alpha_1, \alpha_2, \ldots$ are the positive roots of the transcendental equation,

$$\sin\,(\alpha_n h) = 0, \quad n = 1, 2, \ldots. \tag{21}$$

The transform satisfies the relations,

$$F\left[\frac{\partial^2\theta}{\partial z^2}\right] = -\alpha_n^2\bar{\theta}(\alpha_n, r) \tag{22}$$

$$F\left[\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r}\right] = \frac{\partial^2\bar{\theta}}{\partial r^2} + \frac{1}{r}\frac{\partial\bar{\theta}}{\partial r} \tag{23}$$

Taking the integral transform of the system (Eqs. 15–18) by applying the transform (Eq. 19), one obtains

$$\frac{\partial^2\bar{\theta}}{\partial r^2} + \frac{1}{r}\frac{\partial\bar{\theta}}{\partial r} - \alpha_n^2\bar{\theta} = 0 \tag{24}$$

with

$$\frac{\partial\bar{\theta}}{\partial r} = \sqrt{\frac{2}{h}}\frac{g_0 b}{2k}\frac{[1 - \cos(\alpha_n h)]}{\alpha_n} \quad \text{at } r = b \tag{25}$$

Solution of the differential Eq. 24 is obtained as

$$\bar{\theta}(\alpha_n, r) = \sqrt{\frac{2}{h}}\frac{g_0 b}{2k}\frac{1}{\alpha_n^2}\frac{I_0(\alpha_n r)}{I_1(\alpha_n b)}\,[1 - \cos(\alpha_n h)] \tag{26}$$

On applying the inverse Fourier transform defined in Eq. 20, one obtains

$$\theta(r, z) = \frac{g_0 b}{hk}\sum_{n=1}^{\infty}\frac{1}{\alpha_n^2}\frac{I_0(\alpha_n r)}{I_1(\alpha_n b)}\sin(\alpha_n z)\,[1 - \cos(\alpha_n h)] \tag{27}$$

Substituting Eq. 27 into Eq. 14, one obtains the expression of the temperature distribution function as

$$T(r, z) = \frac{g_0 b}{hk}\sum_{n=1}^{\infty}\frac{1}{\alpha_n^2}\frac{I_0(\alpha_n r)}{I_1(\alpha_n b)}\sin(\alpha_n z)\,[1 - \cos(\alpha_n h)] - \frac{g_0 r^2}{4k} \tag{28}$$

4 Displacement Potential Function and Thermal Stresses

Using Eq. 28 in Eq. 5, one obtains

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} = (1+v)a_t \left[\frac{g_0 b}{hk} \sum_{n=1}^{\infty} \frac{1}{\alpha_n^2} \frac{I_0(\alpha_n r)}{I_1(\alpha_n b)} \sin(\alpha_n z)\left[1-\cos(\alpha_n h)\right] - \frac{g_0 r^2}{4k} \right] \tag{29}$$

Solving Eq. 29, by using the result,

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) I_0(\alpha_n r) = \alpha_n^2 I_0(\alpha_n r),$$

one obtains

$$\frac{\psi}{X} = \left[\frac{g_0 b}{hk} \sum_{n=1}^{\infty} \frac{1}{\alpha_n^4} \frac{I_0(\alpha_n r)}{I_1(\alpha_n b)} \sin(\alpha_n z)\left[1-\cos(\alpha_n h)\right] - \frac{g_0 r^4}{64k} \right] \tag{30}$$

Using Eq. 30 in Eqs. 6 and 7, one obtains the expressions of thermal stresses as

$$\frac{\sigma_{rr}}{Y} = \left[\frac{g_0 b}{hk} \sum_{n=1}^{\infty} \frac{1}{\alpha_n^3} \frac{1}{r} \frac{I_1(\alpha_n r)}{I_1(\alpha_n b)} \sin(\alpha_n z)\left[1-\cos(\alpha_n h)\right] - \frac{1}{16}\frac{g_0 r^2}{k} \right] \tag{31}$$

$$\frac{\sigma_{\theta\theta}}{Y} = -\left[\frac{g_0 b}{hk} \sum_{n=1}^{\infty} \frac{1}{\alpha_n^2} \frac{1}{I_1(\alpha_n b)} \left[\frac{I_1(\alpha_n r)}{\alpha_n r} - I_0(\alpha_n r)\right] \sin(\alpha_n z)\left[1-\cos(\alpha_n h)\right] + \frac{3}{16}\frac{g_0 r^2}{k} \right] \tag{32}$$

where X and Y are the constants and set for convenience, as

$$X = -(1+v)a_t \quad \text{and} \quad Y = 2(1+v)a_t\mu.$$

Also noting that

$$\frac{\partial}{\partial r}(I_0(\alpha_n r)) = \alpha_n I_1(\alpha_n r)$$
$$\frac{\partial^2}{\partial r^2}(I_0(\alpha_n r)) = -\alpha_n^2 \left[\frac{I_1(\alpha_n r)}{\alpha_n r} - I_0(\alpha_n r)\right].$$

5 Numerical Calculations

A numerical calculation has been carried out for an aluminum (pure) circular plate with the following material properties:

Fig. 1 Temperature distribution

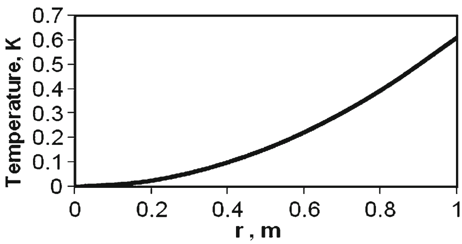
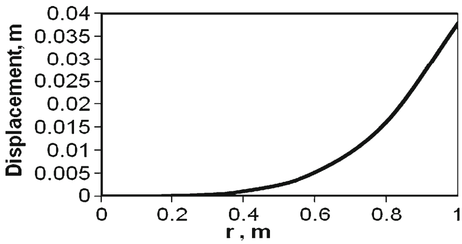


Fig. 2 Displacement function



Material properties and dimensions:

Thermal diffusivity $\alpha = 84.18 \times 10^{-6}\ \mathrm{m^2 \cdot s^{-1}}$
Thermal conductivity $k = 204.2\ \mathrm{W \cdot m^{-1} \cdot K^{-1}}$
Density $\rho = 2707\ \mathrm{kg \cdot m^{-3}}$
Specific heat $c_p = 896\ \mathrm{J \cdot kg^{-1} \cdot K^{-1}}$
Poisson ratio $\nu = 0.35$
Coefficient of linear thermal expansion, $a_t = 23 \times 10^{-6}\ \mathrm{K^{-1}}$
Lamé constant $\mu = 26.67$
Heat source is a constant heat source of strength $g_0 = 50\ \mathrm{W \cdot m^{-3}}$

Dimensions of circular plate

Radius of a thin circular plate $b = 1$ m
Thickness of a thin circular plate $h = 0.1$ m

The numerical calculation has been carried out with the help of computational mathematical software Mathcad-2000, and the graphs are plotted with the help of Excel (MS Office-2007).

As shown in Fig. 1, it can be observed that, due to internal heat generation at a constant rate g_0, the temperature increases from center $r = 0$ of a thin circular plate to the outer circular boundary $r = 1$.

It can be observed (Fig. 2) that the displacement occurs around the outer circular boundary $r = 1$. This displacement is proportional to the temperature.

The radial stress function σ_{rr} (Fig. 3) is zero at the center $r = 0$. As the internal heat source g_0 is constant, we can observe the radial stress function develops tensile stresses within the region $0 \le r \le 1$ and it increases towards the outer circular edge.

The angular stress function $\sigma_{\theta\theta}$ (Fig. 4) increases from the center to the outer circular edge. It is a maximum at the outer circular edge. It develops tensile stresses within the region $0 \le r \le 1$.

Fig. 3 Radial stress function

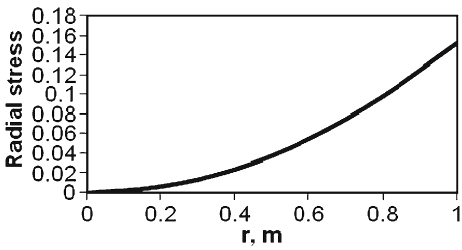
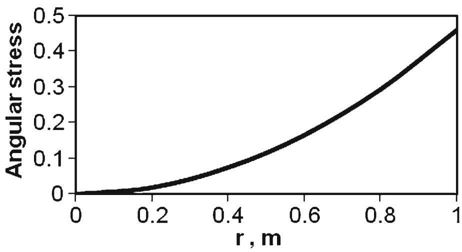


Fig. 4 Angular stress function



6 Discussion of the Results

In this paper we extended the work of Deshmukh and co-worker [3] for a two-dimensional inhomogeneous boundary value problem of heat conduction and determined the expressions for temperature, displacement, and thermal stresses due to a constant internal heat source within it.

As a special case, a mathematical model is constructed for aluminum (pure), a thin circular plate with the material properties specified. The heat source is a constant heat at a rate g_0. We can summarize that due to the small thickness, the stress components and the displacement occur near the heated region. Due to the internal heat generation within the thin circular plate, the radial and angular stresses develop tensile stresses. Also, it is observed from the plots of temperature and displacement, the direction of the heat flow and the direction of body displacement are the same and they are proportional to each other.

The results presented here will be useful in engineering problems, particularly in the determination of the state of stress in a thin circular plate. Also, any particular case of special interest can be derived by assigning suitable values to the parameters and functions in Eqs. 28 and 30–32.

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